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A DERIVED THEORY OF ELASTIC-PLASTIC SHELLS

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Abstract—The key to a theory for elastic-plastic shells is the formulation of constitutive equations. Here, incremental equations are derived from the Hooke, Prandtl-Reuss equations of elastic, plastic deformations. The theory does not embody an initial yield condition, but admits immediate, though gradual, evolution of inelastic strain. Consequently, the abrupt transitions and interfaces between elastic and plastic regions are nonexistant.

Legendre polynomials are employed to approximate the distribution of stresses; the polynomials of first and second degree are identified with the active forces and couples. Higher polynomials represent residual stresses.

The balance of work and rate of dissipation serve to establish the constitutive equations and conditions of loading.

NOTATION

θ^i arbitrary coordinate ($i = 1, 2, 3$); θ^3 denotes distance along normal to reference surface
θ^α arbitrary coordinate of reference surface ($\theta^3 = 0$, $\alpha = 1, 2$)
$\dot{}$ overdot signifies material time derivative or increment
$\mathbf{g}_i, \mathbf{G}_i$ tangent vector of initial, current state
$\mathbf{g}^i, \mathbf{G}^i$ reciprocal vector; $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, $\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j$
g_{ij}, G_{ij} component of covariant metric tensor; $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$
g^{ij}, G^{ij} component of contravariant metric tensor; $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$, $G^{ij} = \mathbf{G}^i \cdot \mathbf{G}^j$
$a_{\alpha\beta}, a^{\alpha\beta}$ component of covariant, contravariant, metric tensor of reference surface ($\theta^3 = 0$); $a_{\alpha\beta} \equiv g_{\alpha\beta}(\theta^1, \theta^2, 0)$, $a^{\alpha\beta} \equiv g^{\alpha\beta}(\theta^1, \theta^2, 0)$
$\sqrt{g}, \sqrt{G}$ metric of initial, current volume; $\sqrt{(g)} \equiv \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)$, $\sqrt{(G)} \equiv \mathbf{G}_1 \cdot (\mathbf{G}_2 \times \mathbf{G}_3)$
$\sqrt{a}, \sqrt{A}$ metric of initial, current area; $a \equiv g(\theta^1, \theta^2, 0)$, $A \equiv G(\theta^1, \theta^2, 0)$
$\bar{\gamma}_{ij}$ component of strain tensor; $\bar{\gamma}_{ij} \equiv \frac{1}{2}(G_{ij} - g_{ij})$
$\mathbf{t}^i$ stress vector (force/initial area) on θ^i surface
s^{ij} component of stress tensor $s^{ij} \equiv \mathbf{t}^i \cdot \mathbf{G}^j \sqrt{(g)}^{(ii)}$
h thickness of shell
s_0, γ_0 yield stress, strain in simple tension
ν Poisson's ratio
z normal coordinate; $z \equiv 2\theta^3/h$
γ_{ij} normalized component of strain; $\gamma_{ij} = \bar{\gamma}_{ij}/\epsilon_0$
σ^{ij} nondimensional component of stress; $\sigma^{ij} = s^{ij}/s_0$
σ second invariant of stress deviator
$C_{\alpha\beta\gamma\eta}$ nondimensional component of flexibility tensor in plane stress; if isotropic, $C_{\alpha\beta\gamma\eta} \equiv (1+\nu)a_{\alpha\gamma}a_{\beta\eta} - \nu a_{\alpha\beta}a_{\gamma\eta}$
P_i Legendre polynomial of degree i
$\epsilon_{\alpha\beta}$ strain of surface; see eqn (7)
$\kappa_{\alpha\beta}$ strain of surface; see eqn (7)
$m_i^{\alpha\beta}$ stress of surface; see eqn (6)
$e^i_{\alpha\beta}$ "elastic" strain of surface; see eqn (11a)
$p^i_{\alpha\beta}$ "plastic" strain of surface; see eqn (11b)

INTRODUCTION

The theory of thin Hookean shells was set forth by Love[1, 2] in 1888 following the earlier work of Aron[3]. The underlying hypothesis asserts that the normal to the reference surface remains normal during deformation. Some discrepancies and apparent inconsistencies have been resolved by the works of contemporary scholars; one can trace the development through the works of Sanders[4], Leonard[5], Koiter[6, 7], Budiansky and Sanders[8], Reissner[9, 10] and Naghdi[11, 12]. The essential features of Love's first-approximation remain: Two symmetric second-order *surface* tensors of strain are expressed linearly in terms of two symmetric *surface* tensors of stress.† Moreover, in the constitutive equations of the thin homogeneous shell, equations relating "extensional" strains and stresses (forces) are uncoupled from equations relating certain "flexural" strains and associated stresses (moments). The theory remains

†If the normal is not constrained to remain normal, then two components of transverse shear strain and two components of shear stress augment the theory of Love.

applicable to finite deflections, buckling and post-buckling of thin shells, if the essential nonlinear terms are retained in the geometrical formulations.

The first-approximation of Love reduces the problem of a thin, but three-dimensional body, to the problem of a surface and to the determination of certain functions in that two-dimensional space. Specifically, the solution determines six stress components, practically speaking, the forces and moments upon a section. A solution can provide no more than these integrals of the stress distribution.

In the quest to bridge the gap between the first-approximation and the theory of the three-dimensional body, several authors[12–16] have proposed theories of higher-order moments. Such a theory of multicouples[16] includes the first-approximation (one-couple theory), but also provides better approximations of the three-dimensional theory (multi-couple theories). The constitutive equations of the elastic shell relate a number of multi-stresses to the same number of multi-strains.

The essential differences between theories of elastic and inelastic shells lie in the constitutive equations. Since the behaviour of the inelastic material depends upon the history of deformation, the constitutive equations can only be linear in the increments of the stresses and strains.

Attempts to develop inelastic theories are few: In 1948 Ilyushin[17] presented stress–strain equations derived from the plasticity theory of Hencky[18] and the kinematical hypothesis of Love. Theories of plasticity are also given by Olszak and Sawczuk[19]. The yield condition of Ilyushin and alternatives[20–25] are discussed by Robinson[26], and the effects of transverse shear stresses are included in the works of Shapiro[21], Haydl and Sherbourne[27] and Robinson[28]. A yield criterion for steel shells is given by Crisfield[29]. Some of the foregoing works consider the evolution of the plastic strains and the evolution of the yield functions during loading; others are primarily concerned with limit analysis. Here, a two-dimensional theory is developed to accommodate arbitrary paths of loading *and* unloading. The formulation provides a first-approximation for elastic-plastic shells.

Earlier investigations[30, 31] suggest that a formulation of the constitutive equations for the shell require higher-order stresses, though not necessarily higher-order strains. In short, the distributions of stress upon the section of the inelastic shell are not adequately described by the usual six components (forces and couples), although the six strains of Love's theory may suffice. Here, the stress and strain distributions through the section are represented by Legendre polynomials[32]; the stresses, and the strains, of the first and second polynomials are identified with the forces and couples, and the extensional and flexural strains, respectively.

Incremental equations are derived from the Prandtl–Reuss[33, 34] equations of elastic-plastic deformations. The underlying theory does not embody an initial yield condition, but admits the immediate though gradual, evolution of inelastic strain according to the endochronic theory of Valanis[35]. A balance of work and energy[36] serves to establish the constitutive equations and the conditions of elastic unloading. The former are coupled linear equations in the increments of the symmetric *surface* tensors of stress and strain.

THEORY OF PLASTICITY

A theory of plasticity is given in recent papers[35, 36] and provides the foundations of our theory of elastic plastic shells. The essential features follow:

The evolution of plastic strain is measured by arc length ζ in a space of strain components. If dilatation is neglected, $\gamma_3{}^3 = -\gamma_\mu{}^\mu$, and the rate of arc length is the invariant $\dot{\zeta}$:

$$\dot{\zeta}^2 = \gamma^{\alpha\beta}\gamma_{\alpha\beta} + \gamma_\alpha{}^\alpha\gamma_\beta{}^\beta. \tag{1}$$

Following Valanis[35], we introduce a "time" λ such that

$$\frac{d\lambda}{d\zeta} > 0 \quad (0 < \zeta).$$

The time λ may also depend upon the state of stress; for our immediate needs, we take

$$\dot{\lambda} = \sqrt{\left(\frac{3}{2}\right)}\, \sigma^{n/2}\dot{\zeta}. \tag{2}$$

Here σ is a second invariant of the stress deviator; in the case of plane stress ($\sigma^{33} = 0$),

$$\sigma \equiv \frac{3}{2}\left(\sigma^{\alpha\beta}\sigma_{\alpha\beta} - \frac{1}{3}\sigma_{\eta}{}^{\eta}\sigma_{\mu}{}^{\mu}\right). \tag{3}$$

Our theory of plasticity and our subsequent developments are not limited to the form (2), however the form is especially useful in initial formulations: It provides an isotropic behavior which approaches ideal plasticity as n increases and, consequently, admits direct comparisons with results of classical ideal plasticity.

Our plane stress–strain relation is similar to the relation of Prandtl–Reuss[33, 34]:

$$\gamma_{\alpha\beta} = \gamma^{E}_{\alpha\beta} + \gamma^{P}_{\alpha\beta} \tag{4a}$$

$$= C_{\alpha\beta\gamma\eta}\dot{\sigma}^{\gamma\eta} + \left(\sigma_{\alpha\beta} - \frac{1}{3}\sigma_{\mu}{}^{\mu}G_{\alpha\beta}\right)\dot{\lambda}. \tag{4b}$$

Here too, our theory is not limited to plane stress ($\sigma^{i3} = 0$), but the latter seems adequate for thin shells and serves to illustrate our formulation.

Our theory[36] admits elastic unloading under a condition of negative dissipation, namely,

$$\dot{\sigma}^{\alpha\beta}\gamma^{P}_{\alpha\beta} < 0. \tag{5}$$

STRESS AND STRAIN DISTRIBUTIONS

One essential difference between the elastic and inelastic shell is the distribution of stress through the thickness. The distribution in the latter is decidedly nonlinear and requires an approximation of higher order. In previous studies[16, 30, 31], higher moments were proposed. Here, we use the Legendre polynomials $P_i(z)$ to represent the stress and strain distributions, and benefit from the orthogonality. The stress distribution is represented by stresses $m_i{}^{\alpha\beta}(\theta^1, \theta^2)$:

$$\sigma^{\alpha\beta} = \sqrt{(1+2i)}\, m_i{}^{\alpha\beta}P_i(z) \tag{6a}$$

$$m_i{}^{\alpha\beta} = \frac{\sqrt{(1+2i)}}{2}\int_{-1}^{1}\sigma^{\alpha\beta}P_i\,\mathrm{d}z. \tag{6b}$$

Summation is implied by the repeated index ($i = 0, 1, \ldots, N$).

In most instances, the kinematical hypothesis of Kirchhoff and Love should suffice to describe the deformations of a thin, albeit inelastic shell. Although refinements are also possible, our present formulation is based upon the simple approximation of strain:

$$\dot{\gamma}_{\alpha\beta} = \dot{\epsilon}_{\alpha\beta}P_0 + \sqrt{(3)}\,\dot{\kappa}_{\alpha\beta}P_1. \tag{7}$$

Practically speaking, the strains $\epsilon_{\alpha\beta}$, and $\kappa_{\alpha\beta}$, are the strains of the reference surface ($z = 0$), and the changes of curvature, respectively.

INCREMENTAL RELATIONS BETWEEN STRESSES AND STRAINS

The power of the stress (per unit of area) is given by the approximation†

$$\dot{w} = \frac{1}{2}\int_{-1}^{1}\sigma^{\alpha\beta}\,\dot{\gamma}_{\alpha\beta}\,\mathrm{d}z. \tag{8a}$$

†Here the curvature is neglected in the integral (8a), as it is in the theories of thin elastic shells. Since eqn (6a) is an approximation, the missing factor ($\sqrt{(g)}/a$) might be incorporated in the left side of (6a).

It follows from (6), (7) and (8a) that

$$\dot{w} = m_0{}^{\alpha\beta}\dot{\epsilon}_{\alpha\beta} + m_1{}^{\alpha\beta}\dot{\kappa}_{\alpha\beta}. \tag{8b}$$

Another form of power is obtained if (4b) is used in (8a) instead of (7); then,

$$\dot{w} = C_{\alpha\beta\gamma\eta}m_i{}^{\alpha\beta}\dot{m}_i{}^{\gamma\eta} + m_i{}^{\alpha\beta}\left(m_{j\alpha\beta} - \frac{1}{3}\, m^{\mu}_{j\mu}a_{\alpha\beta}\right)\dot{f}_{ij} \tag{8c}$$

where

$$\dot{f}_{ij} = \frac{\sqrt{(1+2i)}\sqrt{(1+2j)}}{2}\int_{-1}^{1} P_iP_j\dot{\lambda}\,\mathrm{d}z \tag{9}$$

The balance of (8b) and (8c) is satisfied by the stress–strain equations

$$\dot{\epsilon}_{\alpha\beta} = C_{\alpha\beta\gamma\eta}\dot{m}_0{}^{\gamma\eta} + \left(m_{j\alpha\beta} - \frac{1}{3}\, m^{\mu}_{j\mu}a_{\alpha\beta}\right)\dot{f}_{0j} \tag{10_0}$$

$$\dot{\kappa}_{\alpha\beta} = C_{\alpha\beta\gamma\eta}\dot{m}_1{}^{\gamma\eta} + \left(m_{j\alpha\beta} - \frac{1}{3}\, m^{\mu}_{j\mu}a_{\alpha\beta}\right)\dot{f}_{1j} \tag{10_1}$$

$$0 = C_{\alpha\beta\gamma\eta}\dot{m}_i{}^{\gamma\eta} + \left(m_{j\alpha\beta} - \frac{1}{3}\, m^{\mu}_{j\mu}a_{\alpha\beta}\right)\dot{f}_{ij} \quad i = 2,\ldots,N. \tag{10_i}$$

The first sum on the right of (10_i) is an "elastic" strain $\dot{e}^i_{\alpha\beta}$ and the second is a "plastic" strain $\dot{p}^i_{\alpha\beta}$:

$$\dot{e}^i_{\alpha\beta} \equiv C_{\alpha\beta\gamma\eta}\dot{m}_i{}^{\gamma\eta} \tag{11a}$$

$$\dot{p}^i_{\alpha\beta} \equiv \left(m_{j\alpha\beta} - \frac{1}{3}\, m^{\mu}_{j\mu}a_{\alpha\beta}\right)\dot{f}_{ij}. \tag{11b}$$

In accordance with (5), elastic unloading occurs ($\dot{f}_{ij} = 0$), if

$$\dot{m}_i{}^{\alpha\beta}\dot{p}^i_{\alpha\beta} < 0. \tag{12}$$

COMPUTATION

Our immediate concern is the validation of the constitutive equations (10_i) by comparison with alternatives, in particular, with results of three-dimensional theories. The equations may be tested in various ways: We may prescribe a path of strains ($\dot{\epsilon}_{\alpha\beta}$, $\dot{\kappa}_{\alpha\beta}$) and calculated stresses according to (10_i) or we may prescribe a path of stresses ($\dot{m}_0{}^{\alpha\beta}$, $\dot{m}_1{}^{\alpha\beta}$) and calculate strains. Either calculation generates the nonlinear path via increments. Accuracy depends upon the size of the increments which depend, in turn, upon the prevailing state: For example, a simple tensile path ($\dot{m}_0{}^{11} \neq 0$), or a simple bending path ($\dot{m}_1{}^{11} \neq 0$), approaches a limit point ($\mathrm{d}m_0{}^{11}/\mathrm{d}\epsilon_{11} = 0$, or $\mathrm{d}m_1{}^{11}/\mathrm{d}\kappa_{11} = 0$); obviously the size of the stress increment must diminish and vanish as we approach the limit point. To accommodate such curved paths, we introduce an arc-length s; the increment of length is

$$\dot{s}^2 = \dot{\epsilon}_{\alpha\beta}\dot{\epsilon}^{\alpha\beta} + \dot{\kappa}_{\alpha\beta}\dot{\kappa}^{\alpha\beta} + \dot{m}_i{}^{\alpha\beta}\dot{m}^i_{\alpha\beta}. \tag{13}$$

For a given path, the direction of the strain (or stress) increment is prescribed, but the magnitude of $\dot{s}$ is prescribed rather than the magnitude of the incremental strain (or stress).

The initial step follows the linear equations of elasticity ($\dot{p}^i_{\alpha\beta} = 0$), and eqn (13) determines the magnitude of the strain (or stress) increment after the stress (or strain) components are eliminated by (10_0) and (10_1).

If increments of strain ($\dot{\epsilon}_{\alpha\beta}$, $\dot{\kappa}_{\alpha\beta}$) are prescribed, then each subsequent increment of stress is determined by the inversion of eqns (10_i):

$$\dot{m}_0^{\alpha\beta} = {}^{-1}C^{\alpha\beta\gamma\eta}(\dot{\epsilon}_{\gamma\eta} - \dot{p}^0_{\gamma\eta}) \tag{14$_0$}$$

$$\dot{m}_1^{\alpha\beta} = {}^{-1}C^{\alpha\beta\gamma\eta}(\dot{\kappa}_{\gamma\eta} - \dot{p}^1_{\gamma\eta}) \tag{14$_1$}$$

$$\dot{m}_i^{\alpha\beta} = {}^{-1}C^{\alpha\beta\gamma\eta}(-\dot{p}^i_{\gamma\eta}). \tag{14$_i$}$$

The first approximation of $\dot{p}^i_{\alpha\beta}$ is obtained according to (11b), (9), (2), (1) and (7) wherein the last values of $m^i_{\alpha\beta}$ and σ are employed. Recourse to (13) at each step serves to readjust the strain increments, to maintain the prescribed increment of length $\dot{s}$.

If increments of stress ($\dot{m}_0^{\alpha\beta}$, $\dot{m}_1^{\alpha\beta}$) are prescribed, then the increments of strain ($\dot{\epsilon}_{\alpha\beta}$, $\dot{\kappa}_{\alpha\beta}$) and the remaining increments of stress ($\dot{m}_2^{\alpha\beta}$, $\dot{m}_3^{\alpha\beta}$) are to satisfy (10_i), but the factor $\dot{f}_{ij}$ depends on the strain increments according (9), (2), (1) and (7). Therefore, an initial calculation of each subsequent step utilizes the previous increments of strain to form an initial estimate of the increments $\dot{\zeta}$, $\dot{\lambda}$ and $\dot{f}_{ij}$. The calculation is repeated with successive incremental strains providing a new estimate of the increments $\dot{\zeta}$, $\dot{\lambda}$, $\dot{f}_{ij}$ and $\dot{p}^i_{\alpha\beta}$. During loading, successive increments of plastic strain $\dot{p}^i_{\alpha\beta}$ are expected to increase. Therefore, recourse to (13) is again needed to readjust the stress increments, to maintain the prescribed increments of length $\dot{s}$ and to insure convergence of the iterative process.

SOME RESULTS AND COMPARISONS

Our material is determined by the functional $\lambda(\sigma, \zeta)$. The particular form (2) is convenient because it approaches the form of the ideal elastic-plastic material in the limit, $n \to \infty$. Some stress–strain curves are displayed in Fig. 1 ($n = 1, 2, 5, 10$).

Stress–strain histories can be traced in two ways:

Firstly, we can prescribe a history of strain ($\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$) which determine increments $\dot{\gamma}_{\alpha\beta}$ according to (7). The increments $\dot{\zeta}$ and $\dot{\lambda}$ are then computed by (1) and (2), the increments of the stress components $\dot{\sigma}^{\alpha\beta}$ by (4b) and, finally, the increments of stresses ($m_i^{\alpha\beta}$) are calculated by (6b). These calculations require storage of the stresses $\sigma^{\alpha\beta}$ at numerous stations through the shell ($-1 \le z \le +1$). Here 21 stations are employed. The procedure is essentially an approximation via the three-dimensional theory; applications to practical problems are limited by the storage capacity of the computer:

Secondly, we can prescribe the history of strain ($\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$) and calculate the incremental stresses ($m_i^{\alpha\beta}$) according to our theory and the computational procedures described in the preceding section. Only the stresses $m_i^{\alpha\beta}$ are stored.

Our theory is illustrated by the results of several strain histories: Radial paths and histories of prestretch and pretwist are depicted in Figs. 2 and 3. Figures 4–9 display plots of moment (m_1^{11}) versus curvature (κ_{11}). The same histories were used to obtain curves by the 3-D theory (solid lines) and by our 2-D theory (broken lines). For simplicity, the same Poisson

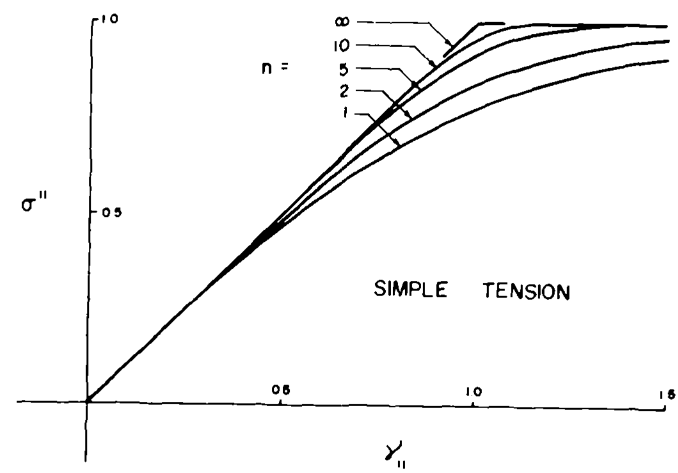

Fig. 1.

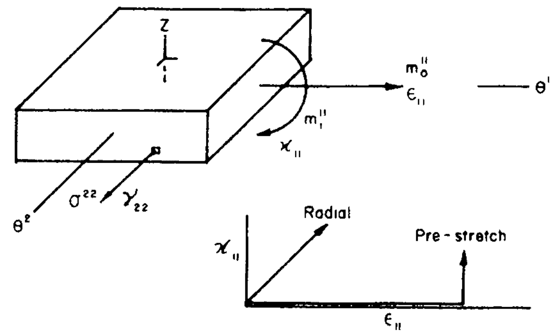

Fig. 2.

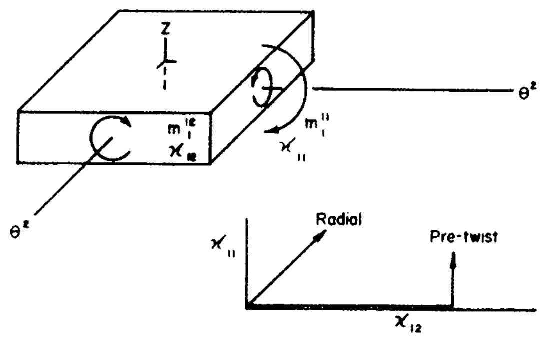

Fig. 3.

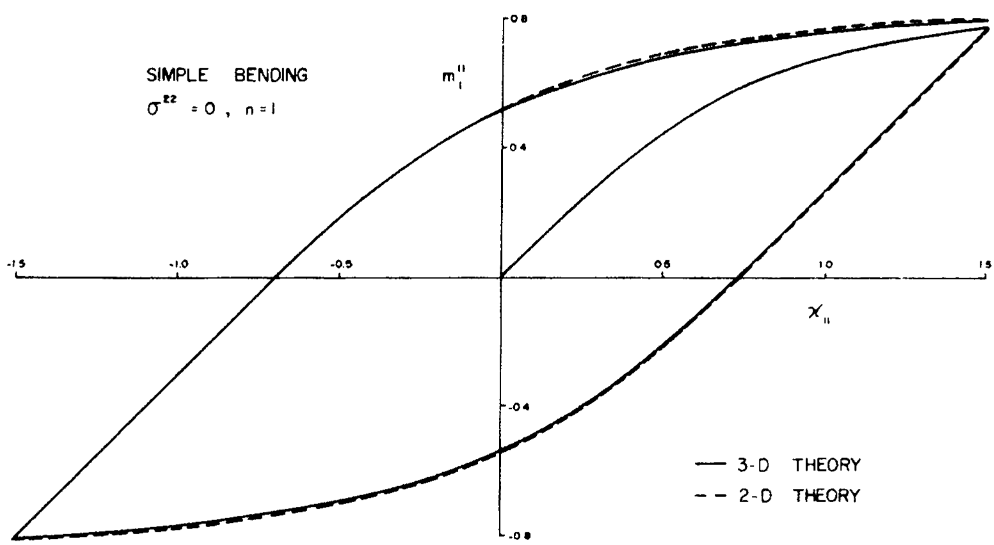

Fig. 4.

ration was used in each case, $\nu = 1/2$. Then the transverse normal stress vanishes ($\sigma^{22} = 0$) in the plots of Figs. 4–6, 8 and 9, wherein $\epsilon_{22} = -\epsilon_{11}/2$, $\kappa_{22} = -\kappa_{11}/2$. Figures 4 and 5 offer a comparison of results for two materials, $n = 1, 10$; for practical purposes, the latter is ideally plastic. Figures 6 and 7 provide a comparison between plane stress $\sigma^{22} = 0$ and plane strain ($\gamma_{22} = \epsilon_{22} = \kappa_{22} = 0$). Figures 8 and 9 show the effects of prestretch and pretwist; a similar effect, a lower stress (m_1^{11}), is evident upon comparisons of the initial loading paths of Figs. 8 or 9 with the path of Fig. 4.

Some discrepancies of the 4-term approximation (broken lines) are evident in Figs. 5 and 7. The former is the curve of a material with a pronounced yielding ($n = 10$) and, consequently, 6

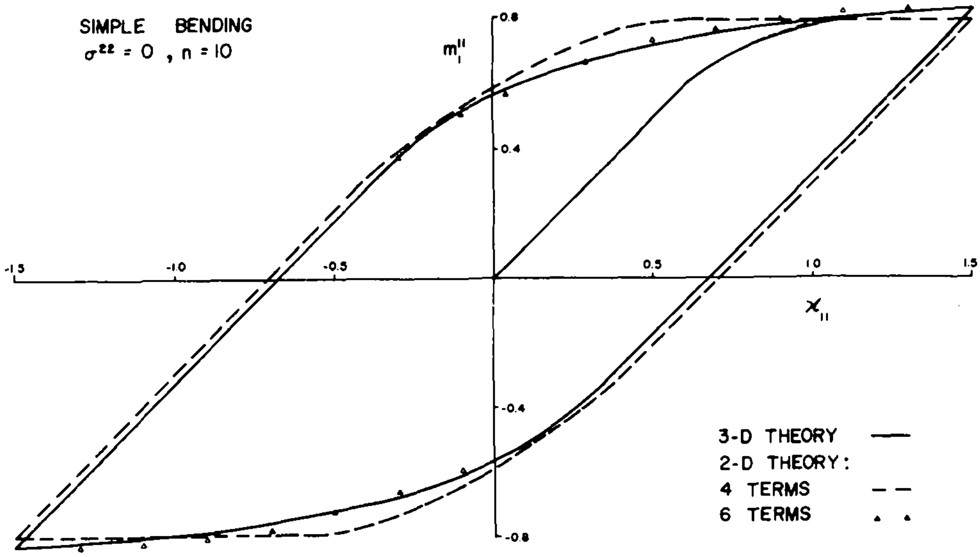

Fig. 5.

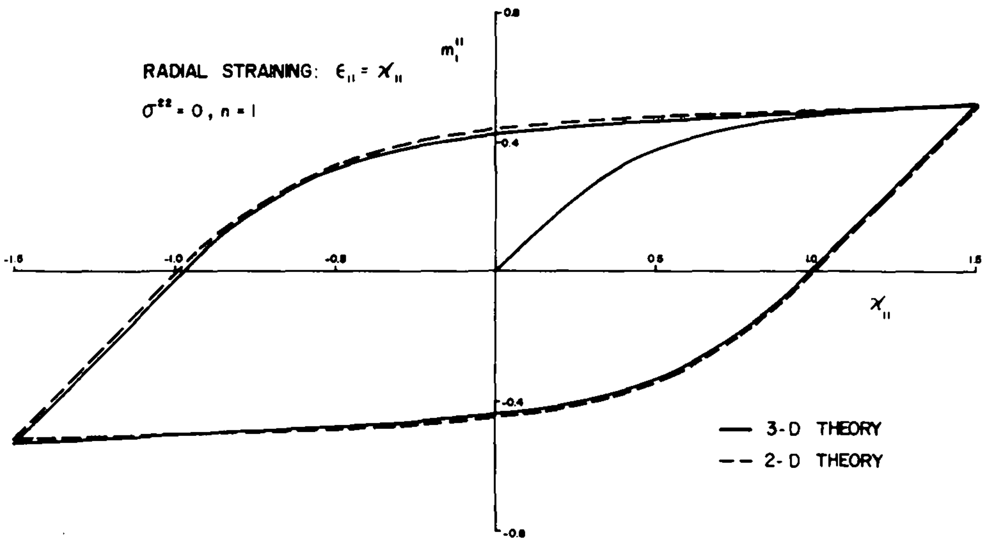

Fig. 6.

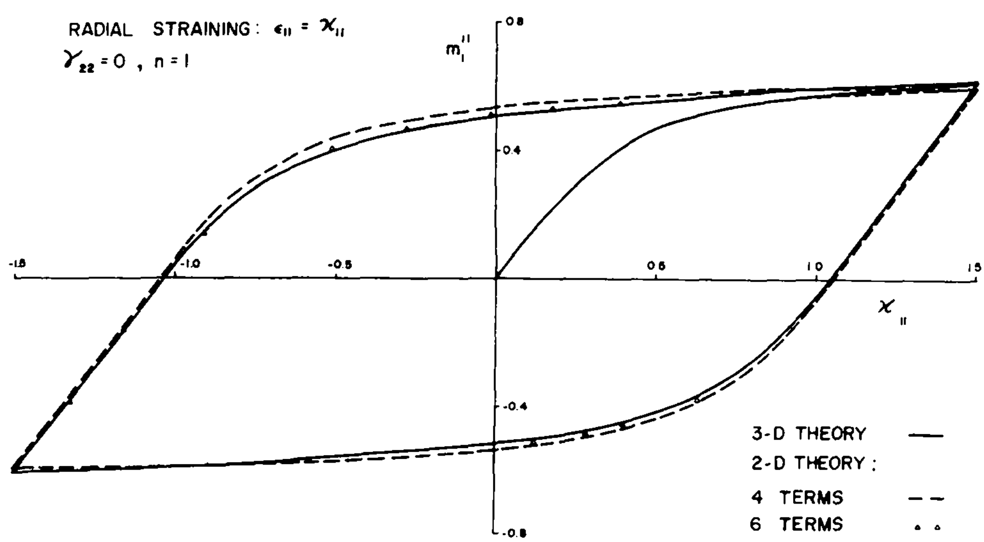

Fig. 7.

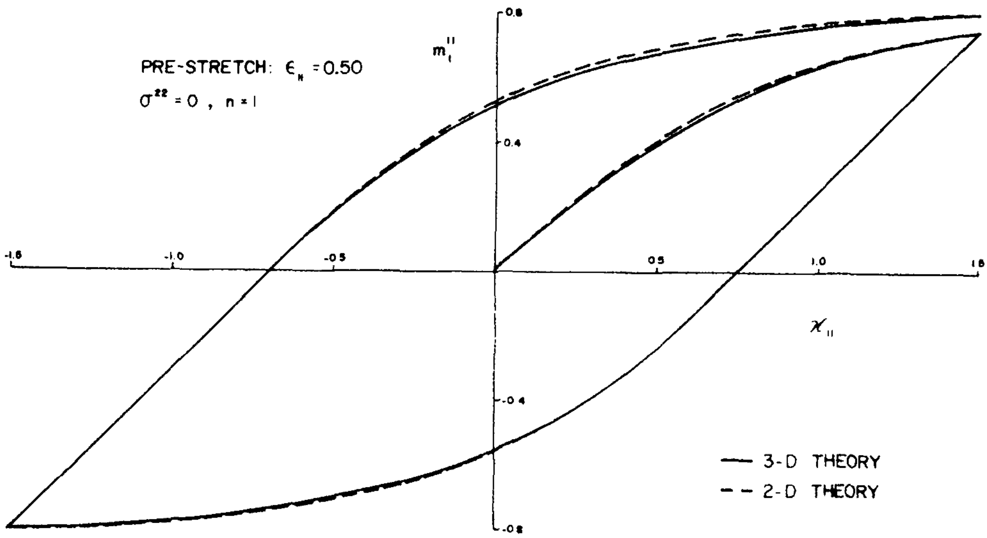

Fig. 8.

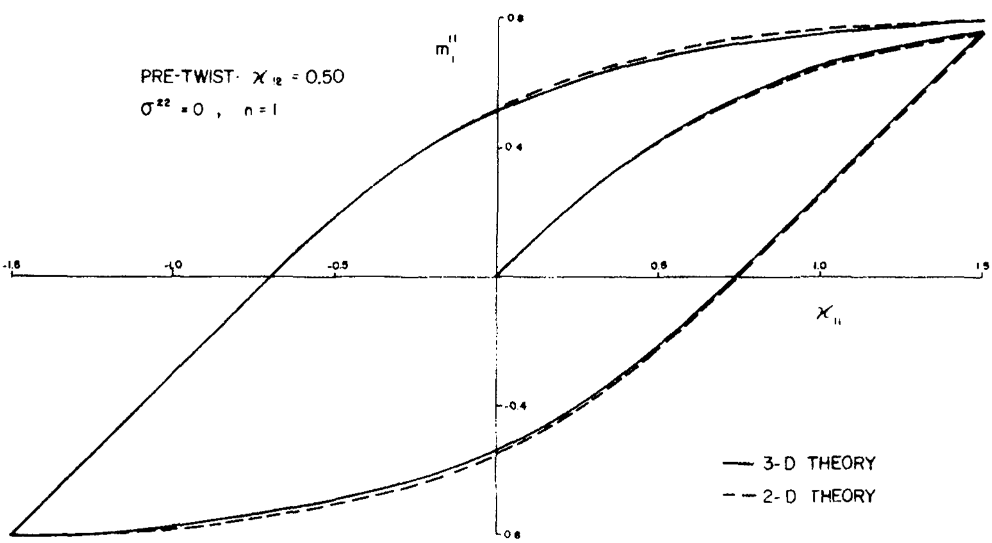

Fig. 9.

terms in eqn (6a) provides a far better approximation of the stress distribution, as indicated by the points of Fig. 5. Also, the greater strains of Fig. 7 introduce discrepancies which are eliminated by the 6-term approximation.

The authors are aware of another theory for homogeneous shells of elastic-plastic material: That theory, developed by M. Bieniek[37], is founded on the concepts of classical plasticity. The initial yield and subsequent yield conditions are expressed as quadratic forms in the stresses $(m_0^{\alpha\beta}, m_1^{\alpha\beta})$, and the increments of plastic strain $(\dot{p}^0_{\alpha\beta}, \dot{p}^1_{\alpha\beta})$ are given by equations of normality (to the yield surface). Comparisons have shown remarkable similarities between the earlier theory[37] and the present theory of $n = 10$.

ON APPLICATION

The present work addresses the central question of elastic-plastic shells; that is the development of the constitutive equations in terms of the two-dimensional fields of stress $(m_i^{\alpha\beta})$ and strain $(\epsilon_{\alpha\beta}, \kappa_{\alpha\beta})$. Such equations are intended for thin shells, wherein the magnitudes of elastic and inelastic strains are comparable. In the analyses of a given shell, these equations must be augmented by kinematical and dynamical equations: The usual equations of thin shells, linear or nonlinear [7], may be employed. In particular, only the customary stresses (forces $m_0^{\alpha\beta}$ and moments $m_1^{\alpha\beta}$) enter the equations of motion, as the additional stresses ($m_2^{\alpha\beta}$, $m_3^{\alpha\beta}$) perform no work upon the strains ($\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$).

The material of our example is described according to eqns (1)–(5) and exhibits the attributes of ideal plasticity ($\sigma \to 1$, $\zeta \to \infty$). Although computations might ultimately lead to limit loads, the theory is not intended as an alternative to the procedures of limit analysis. Indeed, the present theory is better suited to materials which yield gradually and exhibit strain hardening.

CONCLUSIONS

A theory of the elastic-plastic behavior of thin shells is given here in terms of two-dimensional variables of stress ($m_i^{\alpha\beta}$) and strain ($\epsilon_{\alpha\beta}$; $\kappa_{\alpha\beta}$). Although the theory is founded upon certain concepts of endochronic plasticity, yielding does follow the criterion of Mises ($n \to \infty$) and the flow of Prandtl–Reuss.

Numerous comparisons indicate that the theory is a promising basis for approximations of inelastic shells.

Further research is suggested: The general theory can incorporate other materials, additional stresses ($m_i^{\alpha\beta}$), transverse shear stress and strain, and alternatives to the simple unloading condition (12).

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